Performance Analysis of State Dependent Bulk Service Queue with Balking, Reneging and Server Vacation

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Abstract

This investigation deals with state dependent bulk service queue with balking, reneging and multiple vacations where (a-1) arrivals waiting in the queue, server will wait for some time (change over time), in spite of going for a vacation. The server begins service on finding an arrival during the changeover time otherwise the server will go for a vacation. We assume that the late arriving customers in the ongoing service batch if the size of the batch being served is less than a threshold integer c ($a \leq c \leq b$). Performance indices and cost analysis are carried out such as average queue length, expected waiting time in queue and system and variance.

Key Words: Server Vacation, Bulk Service, Balking and Reneging

1. Introduction

Queueing models with server vacations accommodate the real world problems more closely. Such model frequently occurs in transportation systems, computer system, telecommunication, airline scheduling as well as industrial processes such as production/ inventory systems etc. In manufacturing system worker may have some idle time between subsequent jobs. To utilize the time effectively, supervisor can assign another work to the worker. However, it is important that the worker must return to do his primary jobs when he completes the secondary jobs. This motivates us to study such queueing model. Many researchers have paid their attention towards this direction. Boxma and Yechiali [1] studied M/G/1 queueing model with multiple type of feedback and vacation. Gray [2]

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considered vacation queueing model with service interruption. Jayaraman [3] discussed bulk service queueing system with dependent arrival rate on server breakdown. Powell and Humbelt [4] established theoretical and computational analysis for bulk service queue with general control. Reddy and Anitha [5] analyzed stationary distribution of the number of customers in queue and waiting time distribution of an arriving customer for $M/M^{(a,b)}/1$ queueing model with multiple vacations and changeover time. Jain and Jain [7] discussed working vacations model subject to interruption due to multiple types of breakdowns. Jain et al [8] developed a finite queueing model having single and batch service modes for telecommunication system. Chen et al [9] studied a modified Markovian bulk arrival and bulk service queue incorporating state dependent control. Ke [10] analyzed operating characteristic of $M^X/G/1$ queueing system under different vacation policy, where the server leaves for a vacation as soon as the system is empty. Ke et al. [11] considered a finite buffer M/M/C queueing system in which servers are unreliable and follow a (d,c) vacation policy.

The rest of the paper is organized as follow. The model is described in section 2. The mathematical analysis is mentioned in section 3. The steady state equations are obtained in section 4. Other performance indices in terms of probabilities and cost analysis are derived in section 5. The conclusion is given in section 6.

2. Description of the system

Consider $M/M^{(a,c,b)}/1$ queueing system with repeated delayed vacations and changeover time. The basic features of the model are described as follows:

- Customers arrive at the system one by one according to Poisson process with rate λ.
- Service times follow an exponential distribution with mean service rate μ_n
  Where $\mu_n = \begin{cases} \mu, & n = 0 \\ \mu_{1,} & n > 0 \end{cases}$
- The customers are served in batches with quorum level "a" and quota capacity "b".
- If there are i customers ($0 \leq i \leq a - 2$) in the queue, the server will go for a vacation, which is exponentially distributed with parameter β.
- If the server finds a-1 customers in the queue either at a service completion epoch or at a vacation termination point, he will wait for some more time in the system which is called changeover time.
- The changeover time is exponentially distributed with parameter $\propto$.

- When an arrival occurs during this changeover time, the server will start service immediately otherwise at the end of the change over time, the server will go for a vacation.
- On returning to the system after a vacation, if the server finds $0 \leq i \leq a-2$ customers in the queue, the server will go for another vacation and so on until the server, on his arrival finds at least a-1 customers in the queue.
- Vacations, service times and inter-arrival times are mutually independent and identically distributed nonnegative random variables.
- The late arriving customers are allowed to join the batch (without affecting the service time) in case of ongoing service as long as the number of units in that batch is less than 'c'.
- On arrival a customer either decides to join the queue with probability k or balk with probability 1-k.
- Arrival and departure of the impatient customers without getting service are independent. The average reneging rate of the customer can be defined as
  $r(i) = (i-j)\delta, i \geq 0, j = 0, 1, 2$
  Where δ is the rate of time T.

3. Mathematical Analysis

We formulate the process as a continuous time Markov chain with the state space {(i, j)/i≥0,j=0,1}∪{(a-1,2)} where i and j represents the queue size and the state of the server respectively. We assume the following states:

(i) State (i, 0) if there are i customers waiting in the queue and the server is away for vacation.

(ii) State (i, 1) if there are i customers waiting in the queue and the server is busy.

(iii) State (a-1, 2) if there are a-1 customers waiting in the queue and the server is waiting in the system.

We define $P_{i,j}(t)$P = rob. {The system is in state (i,j) at time t}

Let us assume that the steady state probabilities $P_{i,j} = \lim_{t\to\infty} P_{i,j}(t)$ exists

We obtain the following system of equations

$$K\lambda P_{0,0} = \mu P_{0,1} \tag{1}$$

$$K\lambda P_{i,0} = K\lambda P_{i-1,0} + \mu P_{i,1}, \; 1 \leq i \leq a-2 \tag{2}$$

$$(K\lambda+\beta)P_{a-1,0} = K\lambda P_{a-2,0} + \alpha P_{a-1,2} \tag{3}$$

$$(K\lambda+\mu_1)P_{i,0} = K\lambda P_{i-1,0} + \mu P_{i,1},\ a \le i \le d-1 \tag{4}$$

$$(K\lambda+\beta)P_{i,0} = (K\lambda)P_{i-1,0}, \quad i \ge a \tag{5}$$

$$(K\lambda+\mu_1)P_{0,1} = K\lambda P_{a-1,2} + \ \mu\sum_{s=d}^{b} P_{s,1} + \beta\sum_{s=d}^{b} P_{s,0} \tag{6}$$

$$(K\lambda+\mu_1)P_{i,1} = K\lambda P_{i-1,1} + \beta P_{i+b,0} + \mu_1 P_{i+b,1}, i \ge 1 \tag{7}$$

$$(K\lambda+\alpha)P_{a-1,2} = \mu P_{a-1,1} + \beta P_{a-1,0} \tag{8}$$

4. Steady State Probabilities

Equation (7) can be written as

$$[\mu_1 E^{b+1} - (K\lambda+\mu_1)E + K\lambda]P_{i,1} = -\beta P_{i+b+1,0}, i \ge 1 \tag{9}$$

Using the forward shifting operator E defined by $EP_{i,\ 1} = P_{i\ +\ 1,\ 1}$, the characteristic equation of (9) becomes

$$\mu_1 Z^{b+1} - (K\lambda+\mu_1)Z + K\lambda = 0 \tag{10}$$

When $\frac{K\lambda}{b\mu_1}$<1 , Rouche's theorem yields that there exists only one real root of this equation inside the circle $|Z| = 1$. Assuming the stability condition

$\rho = \frac{K\lambda}{b\mu_1} < 1$, if r is the root of the above characteristic equation with $|r| < 1$,

then $$b\rho = \frac{K\lambda}{\mu_1} = \frac{r(1-r^b)}{(1-r)} = r + r^2 + r^3 + \cdots + r^b \tag{11}$$

Using the concept that $P_{i,1} < 1$, the solution of equation (9) is

$$P_{i,1} = \left(Ar^i - B\theta^i\right)P_{a-1,0}, i \ge 0 \tag{12}$$

Where A is an arbitrary constant and

$$B = \frac{\beta\theta^{b-a+1}}{\mu\theta^b - \mu + \beta} \tag{13}$$

Solving equation (8) for $p_{a-1,2}$ and using equation (12) , we get

$$P_{a-1,2} = \frac{1}{(k\lambda+\alpha)}[A\mu r^{a-1} - B\mu\theta^{a-1} + \beta]P_{a-1,0} \tag{14}$$

Summing equation (2) over all relevant i and adding equation (1) , we get

$$k\lambda \sum_{i=0}^{a-2} P_{i,0} = k \sum_{i=0}^{a-3} P_{i,0} + \mu \sum_{i=0}^{a-2} P_{i,1}$$

It follows that

$$P_{a-2,0} = \frac{\mu}{k\lambda}\left[\frac{A\left(1-r^{a-1}\right)}{1-r} - \frac{B\left(1-\theta^{a-1}\right)}{1-\theta}\right] P_{a-1,0},\ 1 \le i \le a-2 \tag{15}$$

Solving equation (2) recursively, we get

$$P_{i,0} = \frac{\mu}{k\lambda}\left[(A-B) + Ar\frac{\left(1-r^{i}\right)}{1-r} - B\theta\frac{\left(1-\theta^{i}\right)}{1-\theta}\right] P_{a-1,0},\ 1 \le i \le a-2 \tag{16}$$

We solve equation (4) recursively and get

$$P_{i,0} = \frac{1}{\gamma^{i+1-a}}\left[1 + \frac{\mu}{k\lambda}\left\{Ar^{a}\left(\frac{1-(\gamma r)^{i+1-a}}{1-\gamma r}\right) - B\theta^{a}\left(\frac{1-(\gamma\theta)^{i+1-a}}{1-\gamma\theta}\right)\right\}\right] P_{a-1,0},\ a \le i \le d-1 \tag{17}$$

Where $\frac{1}{\gamma} = \frac{k\lambda}{k\lambda+\mu_1}$

From equation (5), we get

$$P_{i,0} = \theta^{i+1-a} P_{a-1,0},\ i \ge a-1 \tag{18}$$

Where $\theta = \frac{k\lambda}{k\lambda+\beta}$

Equation (12), (14) and (16-18) gives all the steady state probabilities $P_{i,j}$ in terms of $P_{a-1,0}$ using the normalizing condition

$$\sum_{i=0}^{\infty} P_{i,1} + \sum_{i=0}^{a-1} P_{i,0} + \sum_{i=a}^{d-1} P_{i,0} + P_{a-1,2} = 1 \tag{19}$$

We get

$$P_{a-1,0} = \left[AH(r) - H(\theta) + \frac{\mu}{k\lambda}(A-B)(a-1) + \frac{\gamma^{a-d}-1}{1-\gamma} + \frac{\beta}{k\lambda+\alpha} + 1\right]^{-1} \tag{20}$$

Where

$$H(y) = \frac{1}{1-y} + \frac{\mu y^{a-1}}{k\lambda+\alpha} + \frac{\mu y}{k\lambda(1-y)}\left\{(a-1) - \frac{1-y^{a-1}}{1-y}\right\} + \frac{\mu y^{a}}{k\lambda(1-\gamma y)}\left\{\frac{\gamma^{a-\acute{a}}-1}{1-\gamma} - y\frac{1-y^{\acute{a}-a}}{1-y}\right\} \tag{21}$$

Substituting the probabilities $P_{a-1,0}$,$P_{a-2,0}$ and $P_{a-1,2}$ in equation (3) ,we get

$$A = \frac{\frac{k\lambda(k\lambda+\beta+\alpha)}{k\lambda+\alpha}+\frac{B\mu}{1-\theta}\left[1-\theta^{a-1}\left(\frac{k\lambda+\alpha\theta}{k\lambda+\alpha}\right)\right]}{\left[\mu\left(\frac{1-r^{a-1}}{1-r}\right)+\frac{\alpha}{k\lambda+\alpha}\mu r^{a-1}\right]} \tag{22}$$

5. Performance Measure and Cost Analysis

We discuss some performance measure of the system. Based on these performance measures, we develop a cost model to determine the optimal service rate.

- **Expected queue length L_s in steady state is given by**

$$L_s = \sum_{i=0}^{d-1} iP_{i,0} + \sum_{i=0}^{\infty} iP_{i,1} + (a-1)P_{a-1,2} \tag{23}$$

Substituting for $P_{i,1}$,$P_{a-1,2}$,$P_{i,0}$($1\leq i\leq a-2$) and $P_{i,0}$($a\leq i\leq d-1$) from equations (12),(14),(16) and (17) respectively and simplifying, we get

$$L_s = \left[AF(r) - BF(\theta) + \frac{(a-1)\beta}{k\lambda+\alpha} + \frac{\mu(A-B)(a-2)(a-1)}{2k\lambda} + \frac{a\left(\gamma^{a-d}-1\right)}{1-\gamma} + \frac{\gamma^{d-a-1}-1}{\gamma^{d-a-1}(\gamma-1)^2} - \frac{d-a-1}{\gamma^{d-a}(\gamma-1)}\right] P_{a-1,0} \tag{24}$$

where

$$F(x) = \frac{\mu x^{a-1}(a-1)}{k\lambda+\alpha} + \frac{x}{(1-x)^2} + \frac{\mu x}{k\lambda(1-x)}\left\{\frac{(a-1)(a-2)}{2} - \frac{x\left(1-x^{a-2}\right)}{(1-x)^2} + \frac{(a-2)x^{a-1}}{1-x}\right\} + \frac{x^a\mu}{(1-\gamma x)k\lambda}\left\{\frac{a\left(\gamma^{a-d}-1\right)}{1-\gamma} + \frac{\gamma^{d-a-1}-1}{\gamma^{d-a-1}(\gamma-1)^2} - \frac{d-a-1}{\gamma^{d-a}(\gamma-1)} - x^2\left(\frac{1-x^{d-a-1}}{(1-x)^2}\right) + x\left(\frac{a-(d-1)x^{d-a}}{1-x}\right)\right\} \tag{25}$$

- **Mean number of customer in the queue L_q**

$$L_q = L_s - \left[1 - P_{a-1,0}\right] \tag{26}$$

$$L_q = L_s - \left[\frac{AH(r)-H(\theta)+\frac{\mu}{k\lambda}(A-B)(a-1)+\frac{\gamma^{a-d}-1}{1-\gamma}+\frac{\beta}{k\lambda+\alpha}}{AH(r)-H(\theta)+\frac{\mu}{k\lambda}(A-B)(a-1)+\frac{\gamma^{a-d}-1}{1-\gamma}+\frac{\beta}{k\lambda+\alpha}+1}\right] \tag{27}$$

- **Expected waiting time in the queue W_q**

$$W_q = \frac{L_q}{k\lambda} \tag{28}$$

i.e.

$$W_q = \frac{L_s}{k\lambda} - \frac{1}{k\lambda}\left[\frac{AH(r)-H(\theta)+\frac{\mu}{k\lambda}(A-B)(a-1)+\frac{\gamma^{a-d}-1}{1-\gamma}+\frac{\beta}{k\lambda+\alpha}}{AH(r)-H(\theta)+\frac{\mu}{k\lambda}(A-B)(a-1)+\frac{\gamma^{a-d}-1}{1-\gamma}+\frac{\beta}{k\lambda+\alpha}+1}\right] \quad (29)$$

- **Expected waiting time**

$$W_s = \frac{L_s}{k\lambda} \quad (30)$$

i.e.

$$W_s = \frac{1}{k\lambda}\left[AF(r) - BF(\theta) + \frac{(a-1)\beta}{k\lambda+\alpha} + \frac{\mu(A-B)(a-2)(a-1)}{2k\lambda} + \frac{a(\gamma^{a-d}-1)}{1-\gamma} + \frac{\gamma^{d-a-1}-1}{\gamma^{d-a-1}(\gamma-1)^2} - \frac{d-a-1}{\gamma^{d-a}(\gamma-1)}\right] P_{a-1,0} \quad (31)$$

- **Variance of queue length**

$$Var(i) = \left[\sum_{i=0}^{d-1} i^2 P_{i,0} + \sum_{i=0}^{\infty} i^2 P_{i,1} + (a-1)^2 P_{a-1,2}\right] - L_s^{\,2} \quad (32)$$

- **Busy probability of the server P_B**

$$P_B = P_{a-1,2} + \sum_{n=a-1}^{\infty} P_{i,1} \quad (33)$$

- **Vacation probability of the server P_V**

$$P_V = 1 - P_B \quad (34)$$

- **Expected number of the waiting customers $E(N_q)$**

$$E(N_q) = \sum_{i=0}^{a-2} iP_{i,0} + (i-2)P_{a-1,2} + \sum_{i=a}^{\infty} (i-1)P_{i,1} \quad (35)$$

- **Expected number of the customers in the system E(N)**

$$E(N) = (a-1)P_{a-1,2} + \sum_{i=0}^{a-2} iP_{i,0} + \sum_{i=a}^{\infty} iP_{i,1} \quad (36)$$

- **The average balking rate B.R.**

$$B.R. = [\lambda(1-k)]\left[\sum_{i=0}^{a-2} P_{i,0} + P_{a-1,2} + \sum_{n=a-1}^{\infty} P_{i,1}\right] \quad (37)$$

- **The average reneging rate R.R.**

$$R.R. = \sum_{i=0}^{a-2} i\delta P_{i,0} + [(a-1)-2]\delta P_{a-1,2} + \sum_{i=a}^{\infty}(i-1)\delta P_{i,1} \quad (38)$$

- **The average rate L.R. of customer loss**

L.R. = Balking Rate +Reneging Rate

Cost Model

We develop an expected cost model, in which service rate μ is the control variable. Our aim is to control the service rate to minimize the system's total average cost per unit. Let

C_1= cost per unit time when the server is busy,

C_2= cost per unit time when the server is on vacation,

C_3= cost per unit time when a customer joins in the queue and waits for service,

C_4= cost per unit time when a customer balks or reneges.

The total expected cost function per unit time is given by

$$F(\mu) = C_1 P_B + C_2 P_V + C_3 E(N_q) + C_4 L.R.$$

6. Conclusion

In this paper ,we obtained the general bulk service queueing system with departure of the impatient customers and server vacations. We developed the equations of the steady state probabilities along with some performance measures of the system. We also formulate the cost model to determine the optimal service rate.

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